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Consequences of dynamic yield surface in viscoplasticity

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Abstract

A theory of viscoplasticity is formulated within a thermodynamic concept. The key point is the postulate of a dynamic yield surface, which allows us to take advantage of the postulate of maximum dissipation to derive an associated formulation of the evolutions laws for the internal variables without using penalty techniques that only hold in the limit it when viscoplasticity degenerates to inviscid plasticity. Even a non-associated formulation is presented. Within this general formulation, a particular format of the dynamic yield function enables us to derive the static yield function in a consistent manner. Hardening, perfect and softening viscoplasticity is also defined in a consistent manner. The approach even includes associated and non-associated viscoplasticity where corners exist on the yield and potential surfaces.

Keywords: Thermodynamic of solids; Viscoplasticity; Creep; Flow-rule

1. Introduction

Today there exist two major concepts when formulating viscoplastic models: the Perzyna or the Duvaut–Lions format. In the Perzyna model, Perzyna (1971), the direction of viscoplastic flow is in general determined by the gradient of a plastic potential function calculated at the current stress point. In the Duvaut–Lions model, Duvaut and Lions (1962), the concept of closest-point projection of the stress onto a static yield surface is introduced and the direction of viscoplastic flow is then determined by the difference between the current stress and the closest-point projection. Here, we shall concentrate on the Perzyna formulation. A thermodynamic formulation of the Duvaut–Lions model based on the concept of additive split of the conjugated forces has recently been presented by Ristinmaa and Ottosen (1998).

The postulate of maximum dissipation plays an important role in the thermodynamic treatment of

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inviscid plasticity and it leads to the associated plasticity formulation. This postulate is not a principle in the sense of being a law of nature; instead it may be viewed simply as a convenient mathematical means to fulfil the dissipation inequality. For Perzyna viscoplasticity, however, use of the postulate of maximum dissipation has been restricted to rely on subtle regularization and penalty techniques, that only hold in the limit when viscoplasticity degenerates to inviscid plasticity, e.g. Simo and Honein (1990). Here we will introduce the concept of a dynamic yield surface and this will allow us to use the postulate of maximum dissipation in a straightforward manner. Within this general formulation, a particular format of the dynamic yield function enables us to derive the static yield function in a consistent manner. This approach is then used to derive some of the well-known viscoplastic models and it is shown that the Perzyna model can be given both an associated and a non-associated formulation. Non-associated viscoplasticity follows when the postulate of maximum dissipation cannot be used. It is also shown that a model with no elastic region fits into the formulation proposed and, as an example, Odquist creep is derived.

The viscoplastic formulation proposed is shown to contain inviscid plasticity as a special case when the total strain rate becomes infinitely small and hardening, perfect and softening viscoplasticity is defined in a consistent manner.

In the case of corner viscoplasticity, Mroz and Sharma (1980) present some of the different methods that can be used to handle corner viscoplasticity, although they focused on numerical issues. Here, we shall also present a consistent thermodynamic theory for corner viscoplasticity and both associated viscoplasticity and the non-associated case where the number of yield and potential surfaces may differ are treated.

It was argued by Simo et al. (1988) that Perzyna corner viscoplasticity in general, is not well defined since it in the limit does not reduce to proper corner inviscid plasticity. This argument is based on the idea that when recovering inviscid plasticity at a corner, all yield surfaces are active during this process. However, we shall show that this conception is incorrect.

2. Thermodynamic basis

The assumption of small strains is made. This allows a decomposition of the total strain tensor into an elastic part $\epsilon_{ij}^{\mathrm{e}}$ and a viscoplastic part $\epsilon_{ij}^{\mathrm{vp}}$, i.e.

$$\epsilon_{ij} = \epsilon_{ij}^{\mathrm{e}} + \epsilon_{ij}^{\mathrm{vp}} \tag{1}$$

With θ being the absolute temperature, let us consider the following form of Helmholtz's free energy function ψ per unit volume

$$\psi = \psi^{\mathrm{e}}(\epsilon_{ij} - \epsilon_{ij}^{\mathrm{vp}}, \theta) + \psi^{\mathrm{p}}(\kappa_\alpha, \theta) \tag{2}$$

where κ_α denotes a set of viscoplastic variables, i.e. internal variables, which may be scalars or second-order tensors. Moreover, the number of internal variables may be one, two or more and this is indicated by the Greek subscript α. The decomposition (2) corresponds to the assumption that the instantaneous elastic response does not depend on the internal variables κ_α, cf Lubliner (1972). This decomposition is not necessary, but it has been chosen as it facilitates the exposition. With s being the entropy per unit volume and σ_{ij} the stress tensor, and Clausius–Duhem inequality then takes the form

$$-\dot{\psi} - \dot{\theta} s + \sigma_{ij}\dot{\epsilon}_{ij} \geq 0 \tag{3}$$

for any admissible process; here a dot denotes the rate with respect to time. Whereas (3) expresses the non-negative mechanical entropy production, the non-negative thermal entropy production is expressed by $-q_i\theta_{,i}/\theta \geq 0$ where q_i is the heat flux vector. As usual, this latter inequality is fulfilled trivially by relating q_i to the temperature gradient θ_i via Fourier's law. Taking the rate of (2) and substituting into (3), we obtain that an allowable solution is given by

$$\sigma_{ij} = \frac{\partial \psi^{\mathrm{e}}}{\partial \epsilon_{ij}}; \quad s = -\frac{\partial \psi}{\partial \theta} = -\left(\frac{\partial \psi^{\mathrm{e}}}{\partial \theta} + \frac{\partial \psi^{\mathrm{p}}}{\partial \theta}\right) \tag{4}$$

and the dissipation inequality

$$\gamma \equiv -\frac{\partial \psi^{\mathrm{e}}}{\partial \epsilon_{ij}^{\mathrm{vp}}}\dot{\epsilon}_{ij}^{\mathrm{vp}} - \frac{\partial \psi^{\mathrm{p}}}{\partial \kappa_\alpha}\dot{\kappa}_\alpha \geq 0 \tag{5}$$

Define the thermodynamic forces $\sigma_{ij}^{\mathrm{vp}}$ conjugated to the flux $\dot{\epsilon}_{ij}^{\mathrm{vp}}$ and the thermodynamic forces K_α conjugated to the fluxes $\dot{\kappa}_\alpha$ by

$$\sigma_{ij}^{\mathrm{vp}} = -\frac{\partial \psi^{\mathrm{e}}}{\partial \epsilon_{ij}^{\mathrm{vp}}}; \quad K_\alpha = -\frac{\partial \psi^{\mathrm{p}}}{\partial \kappa_\alpha} \tag{6}$$

Since $\partial\psi^{\mathrm{e}}/\partial\epsilon_{ij}^{\mathrm{vp}} = -\partial\psi^{\mathrm{e}}/\partial\epsilon_{ij}$, it appears that

$$\sigma_{ij}^{\mathrm{vp}} = \sigma_{ij} \tag{7}$$

i.e., the dissipation inequality takes the form

$$\gamma \equiv \sigma_{ij}\dot{\epsilon}_{ij}^{\mathrm{vp}} + K_\alpha\dot{\kappa}_\alpha \geq 0 \tag{8}$$

Differentiation of (4a) then gives the rate of stress tensor

$$\dot{\sigma}_{ij} = D_{ijkl}(\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^{\mathrm{vp}}) + \frac{\partial^2 \psi^{\mathrm{e}}}{\partial \epsilon_{ij}\,\partial \theta}\dot{\theta} \quad \text{where } D_{ijkl} = \frac{\partial^2 \psi^{\mathrm{e}}}{\partial \epsilon_{ij}\,\partial \epsilon_{kl}} \tag{9}$$

We note that the elastic stiffness tensor D_{ijkl} is symmetric and in general not constant. However, we shall assume D_{ijkl} to depend only on the temperature θ, i.e. $D_{ijkl} = D_{ijkl}(\theta)$. The rate of the thermodynamic conjugated forces K_α is obtained by differentiation of (6b), i.e.

$$\dot{K}_\alpha = -D^*_{\alpha\beta}\dot{\kappa}_\beta - \frac{\partial^2 \psi^{\mathrm{p}}}{\partial \kappa_\alpha\,\partial \theta}\dot{\theta} \tag{10}$$

where the symmetric tensor $D^*_{\alpha\beta}$ is defined by

$$D^*_{\alpha\beta} = \frac{\partial^2 \psi^{\mathrm{p}}}{\partial \kappa_\alpha\,\partial \kappa_\beta} \tag{11}$$

In general, we have $D^*_{\alpha\beta} = D^*_{\alpha\beta}(\kappa_\gamma, \theta)$.

3. Dynamic yield function — evolution laws

From the thermodynamic formulation, the constitutive laws for the stress tensor σ_{ij} (4a) and the thermodynamic forces K_α (6b) were obtained, but no information is given about the evolution laws for the viscoplastic strains and the internal variables. The only restriction on these evolution laws is that the second law of thermodynamics must be fulfilled, i.e. that the dissipation inequality (8) must be fulfilled.

However, before these evolution laws can be established, we have to define a condition, which enables us to determine whether viscoplastic behaviour occurs or purely elastic behaviour occurs. For this purpose, we assume the existence of the following function

$$f = f(\sigma_{ij}, K_\alpha, \dot{\epsilon}_{ij}^{\mathrm{vp}}, \dot{\kappa}_\alpha, \theta) \tag{12}$$

This means that f depends on the fluxes $\dot{\epsilon}_{ij}^{\mathrm{vp}}$ and $\dot{\kappa}_\alpha$, the temperature θ and the corresponding conjugated forces σ_{ij} and K_α, cf (8). The possible influence of temperature is also reflected by $K_\alpha = K_\alpha(\kappa_\beta, \theta)$ as apparent from (6) and (2). In order to detect whether viscoplastic loading occurs or elastic unloading occurs, we make the following definition

$$f = 0 \Leftrightarrow \text{possibility for viscoplastic response}$$

$$f < 0 \Leftrightarrow \text{elastic response} \tag{13}$$

and the condition $f > 0$ is not allowed. It appears that the expression $f = 0$ corresponds to what has been termed the dynamic yield surface, cf Perzyna (1971) and Phillips and Wu (1973); however, contrary to the present approach these authors did not pursue the implications of this concept. The dynamic yield surface differs from the so-called static yield surface to be introduced later.

With these definitions, the evolution laws can now be derived by making use of the postulate of maximum dissipation. We are then faced with the following problem: for given fluxes $\dot{\epsilon}_{ij}^{\mathrm{vp}}$ and $\dot{\kappa}_\alpha$ and temperature θ, find those stresses σ_{ij} and forces K_α that minimize the quantity $-\gamma$, where γ is given by (8), under the constraint $f \leq 0$, cf (13). Following, for instance, Luenberger (1984) p. 314, we are then led to the following evolution laws

$$\dot{\epsilon}_{ij}^{\mathrm{vp}} = \Lambda \frac{\partial f}{\partial \sigma_{ij}}; \quad \dot{\kappa}_\alpha = \Lambda \frac{\partial f}{\partial K_\alpha} \tag{14}$$

and the Kuhn–Tucker relations given by

$$\Lambda \geq 0, \quad f \leq 0 \quad \text{and} \quad \Lambda f = 0 \tag{15}$$

For given temperature and fluxes $\dot{\epsilon}_{ij}^{\mathrm{vp}}$ and $\dot{\kappa}_\alpha$, f is required to be a convex and smooth function. The evolution laws derived correspond to associated viscoplasticity and they certainly fulfil the dissipation inequality. In view of (12), we may note that, in general, (14) are implicit equations in the fluxes $\dot{\epsilon}_{ij}^{\mathrm{vp}}$ and $\dot{\kappa}_\alpha$.

It turns out to be possible to obtain a more general format, namely non-associated viscoplasticity. Traditionally, the function f in (14) is then replaced by a potential function g that depend on the same variables as f, i.e.

$$g = g(\sigma_{ij}, K_\alpha, \dot{\epsilon}_{ij}^{\mathrm{vp}}, \dot{\kappa}_\alpha, \theta) \tag{16}$$

cf (12). The following evolution laws are then postulated

$$\dot{\epsilon}_{ij}^{\mathrm{vp}} = \Lambda \frac{\partial g}{\partial \sigma_{ij}}$$

$$\dot{\kappa}_\alpha = \Lambda \frac{\partial g}{\partial K_\alpha} \quad \Lambda \geq 0, \quad f \leq 0 \quad \text{and} \quad \Lambda f = 0 \tag{17}$$

It appears that the non-associated formulation reduces to the associated formulation when $g = f$. However, whereas the associated formulation fulfils the dissipation inequality (8) just by requiring f to be a convex function, it is more difficult to ensure that the non-associated formulation (17) fulfils the dissipation inequality. To illustrate an allowable formulation in terms of (17) consider the following situation: for given fluxes and temperature, let the potential function be smooth and convex in the σ_{ij}, K_α-space. The expression, $g = C$ = constant then describes a surface in that space. If the value of g at the origin of the space is less than C = constant, then it follows directly that formulation (17) fulfils the dissipation inequality (8), cf Eringen (1975).

4. Introduction of the static yield function

It turns out to be advantageous to simplify the general framework described above. For that purpose, we choose a function $\bar{F}$ that possesses the following properties

$$\bar{F} = \bar{F}(\sigma_{ij}, K_\alpha, \dot{\epsilon}_{ij}^{\text{vp}}, \dot{\kappa}_\alpha, \theta) \geq 0 \tag{18}$$

and

$$\bar{F}(\sigma_{ij}, K_\alpha, \dot{\epsilon}_{ij}^{\text{vp}} = 0, \dot{\kappa}_\alpha = 0, \theta) = 0; \quad \bar{F}(\sigma_{ij}, K_\alpha, \dot{\epsilon}_{ij}^{\text{vp}} \neq 0, \dot{\kappa}_\alpha \neq 0, \theta) > 0 \tag{19}$$

Moreover, we choose a function $F = F(\sigma_{ij}, K_\alpha, \theta)$ and express the dynamic yield function f as

$$f(\sigma_{ij}, K_\alpha, \dot{\epsilon}_{ij}^{\text{vp}}, \dot{\kappa}_\alpha, \theta) = F(\sigma_{ij}, K_\alpha, \theta) - \bar{F}(\sigma_{ij}, K_\alpha, \dot{\epsilon}_{ij}^{\text{vp}}, \dot{\kappa}_\alpha, \theta) \tag{20}$$

From (19) and (20), it then follows that

$$f(\sigma_{ij}, K_\alpha, \dot{\epsilon}_{ij}^{\text{vp}} = 0, \dot{\kappa}_\alpha = 0, \theta) = F(\sigma_{ij}, K_\alpha, \theta) \tag{21}$$

Moreover, (18) and (20) results in

$$f \leq F \tag{22}$$

The format of the dynamic yield function given by (20) presents a more restricted group of materials than those given by (12), but it will turn out that this restricted group of materials is sufficiently general to encompass a number of well-known viscoplasticity formulations. In particular, it allows us to derive the static yield surface in a consistent manner, as will be shown next.

The conditions for viscoplastic response or elastic response given by (13) and the Kuhn–Tucker relations (17) are expressed in terms of the dynamic yield function f and we shall now show that these conditions can be expressed in terms of the function F.

Suppose first that $F < 0$; from (22) it follows that $f < 0$, i.e. (17) implies $\Lambda = 0$. Suppose next that $F = 0$; in accordance with (20) we then have $f = -\bar{F}$. If $\Lambda > 0$ then (17) shows that $\dot{\epsilon}_{ij}^{\text{vp}} \neq 0$ and $\dot{\kappa}_\alpha \neq 0$, i.e. (19) shows that $\bar{F} > 0$ and $f = -\bar{F}$ then implies $f < 0$, which, according to (17), results in $\Lambda = 0$, i.e. we have obtained a contradiction. If $\Lambda = 0$ (and still assuming $F = 0$ and thereby $f = -\bar{F}$), we obtain from (19) that $\bar{F} = 0$ and thereby $f = 0$, which according to (17) is acceptable. These remarks lead to the observation that $F \leq 0 \Rightarrow \Lambda = 0$. Suppose next that $\Lambda = 0$; then (19) implies $\bar{F} = 0$ and (20) then gives $f = F$. Since $f \leq 0$, we obtain $F \leq 0$. Summarizing, we have

$$F \leq 0 \Leftrightarrow \Lambda = 0 \tag{23}$$

We are then left with

$$F > 0 \Leftrightarrow \Lambda > 0 \tag{24}$$

With (23) and (24) we are now able to evaluate whether viscoplastic or elastic loading occurs just by evaluating the sign of the function $F(\sigma_{ij}, K_\alpha, \theta)$. Moreover, while $f = 0$ holds during viscoplastic loading, it follows from (22) that the surface $f = 0$ will contain the surface $F = 0$.

Assume that the thermodynamic forces K_α and temperature θ are given and fixed. From (23) and (24), we found that if the stress state is located inside the surface $F = 0$ then an elastic response occurs whereas a stress state outside the surface $F = 0$ implies development of viscoplasticity. In the limit when the surface $F = 0$ is approached from outside, then the viscoplastic effects approach zero.

It follows that the surface $F = 0$ describes the so-called static yield surface and in the expression $F(\sigma_{ij}, K_\alpha, \theta)$, the forces K_α describe the hardening parameters. Moreover, it is recalled that the dynamic yield surface $f = 0$ passes through the current stress point and is obtained by an enlargement of the static yield surface. We have then shown that by taking the dynamic yield function f in the format (20), the static yield function F emerges in a natural manner.

The multiplier Λ enters the evolution laws (17) and it still remains to obtain an expression that determines this quantity. In the usual Perzyna formulation, Λ is simply chosen to be any non-negative quantity. Here, however, we have a priori required $f = 0$ to hold during viscoplasticity and since f depends on the fluxes $\dot{\epsilon}_{ij}^{\mathrm{vp}}$ and $\dot{\kappa}_\alpha$, which, in turn, depends on Λ, the function f and Λ must be related. To identify this relation as simply as possible, we shall assume that the potential function g given by (16) can be written in the following simplified form

$$g = G(\sigma_{ij}, K_\alpha, \theta) \tag{25}$$

This simplified form turns out to be sufficiently general to encompass most well-known viscoplasticity formulations. It follows that the evolution Eqs. (17) now take the form

$$\dot{\epsilon}_{ij}^{\mathrm{vp}} = \Lambda \frac{\partial G}{\partial \sigma_{ij}}; \quad \dot{\kappa}_\alpha = \Lambda \frac{\partial G}{\partial K_\alpha} \tag{26}$$

During viscoplastic loading, we have $f = 0$ and (20) then gives

$$F(\sigma_{ij}, K_\alpha, \theta) = \bar{F}(\sigma_{ij}, K_\alpha, \dot{\epsilon}_{ij}^{\mathrm{vp}}, \dot{\kappa}_\alpha, \theta) \tag{27}$$

Insertion of (26) results in

$$F(\sigma_{ij}, K_\alpha, \theta) = \bar{F}\left(\sigma_{ij}, K_\alpha, \Lambda \frac{\partial G}{\partial \sigma_{ij}}, \Lambda \frac{\partial G}{\partial K_\alpha, \theta}\right) \quad \text{during viscoplastic loading} \tag{28}$$

It appears that if the functions F and $\bar{F}$ are known then this expression enables us to determine the multiplier Λ; we shall later see examples of this approach.

We found that the multiplier Λ can be determined from (28), but that this requires the potential function g to be given in the form (25). Let us now investigate the consequences of (25) for associated viscoplasticity. Associated viscoplasticity requires that $\partial g/\partial \sigma_{ij} = \partial f/\partial \sigma_{ij}$ and $\partial g/\partial K_\alpha = \partial f/\partial K_\alpha$; with (25) and (20) these requirements become $\partial G/\partial \sigma_{ij} = \partial F/\partial \sigma_{ij} - \partial \bar{F}/\partial \sigma_{ij}$ and $\partial G/\partial K_\alpha = \partial F/\partial K_\alpha - \partial \bar{F}/\partial K_\alpha$. Since $G(\sigma_{ij}, K_\alpha, \theta)$, $F(\sigma_{ij}, K_\alpha, \theta)$ and $\bar{F}(\sigma_{ij}, K_\alpha, \dot{\epsilon}_{ij}^{\mathrm{vp}}, \dot{\kappa}_\alpha, \theta)$, these requirements can only be fulfilled if $\partial \bar{F}/\partial \sigma_{ij} = 0$ and $\partial \bar{F}/\partial K_\alpha = 0$. If follows that

$$\text{Associated viscoplasticity requires:} \quad \bar{F} = \bar{F}(\dot{\epsilon}_{ij}^{\text{vp}}, \dot{\kappa}_\alpha, \theta) \quad \text{and} \quad G(\sigma_{ij}, K_\alpha, \theta) = F(\sigma_{ij}, K_\alpha, \theta)$$
$$\Longrightarrow \dot{\epsilon}_{ij}^{\text{vp}} = \Lambda \frac{\partial f}{\partial \sigma_{ij}} = \Lambda \frac{\partial F}{\partial \sigma_{ij}}; \quad \dot{\kappa}_\alpha = \Lambda \frac{\partial f}{\partial K_\alpha} = \Lambda \frac{\partial F}{\partial K_\alpha} \tag{29}$$

With these remarks, let us return to the thermodynamic formulation and in particular the evolution laws and fulfilment of the dissipation inequality (8). For associated viscoplasticity, these evolution laws follow from the postulate of maximum dissipation and they are given by (14) in the general case and by (29) for the more restricted group of materials defined by (20) and (25). The postulate of maximum dissipation in combination with the dynamic yield function f being convex and smooth ensures that the dissipation inequality is fulfilled. When the dynamic yield function is given by (20) and associated viscoplasticity is considered, cf (29), the requirements of convexity and smoothness of f becomes convexity and smoothness of the static yield function F.

It appears that we have obtained a convenient thermodynamic formulation of associated viscoplasticity without having to rely on subtle regularization or penalty techniques, cf Simo and Honein (1990), that only holds for the Perzyna formulation in the limit when viscoplasticity degenerates to inviscid plasticity. Instead, the present formulation only hinges on the concept of a dynamic yield function, which enables us to make use of the postulate of maximum dissipation. Moreover, by suitable choices of the involved functions the concept of a static yield function emerges in a natural manner.

For non-associated plasticity, the evolution laws are in the general case given by (17) and they fulfil the dissipation inequality (8) if the potential function fulfils the requirements stated in the discussion following (17). For the more restricted group of materials defined by (20) and (25), the evolution laws are given by (26) and they fulfil the dissipation inequality if: (1) G is a smooth and convex function in the σ_{ij}, K_α-space; the expression $G = C$ = constant then describes a surface in that space; (2) the value of G at the origin of the space is less than C = constant.

To substantiate the type of viscoplasticity defined by (20) and (25), we shall later show that it contains a variety of previously proposed viscoplastic theories as special cases. Before that, we shall discuss the relation between viscoplasticity and inviscid plasticity.

5. The limit case of inviscid plasticity

It turns out that the viscoplastic formulation defined by (20) and (25) reduces in the limit to classical inviscid plasticity. To achieve this, some preliminary derivations are necessary. For simplicity, isothermal conditions will be assumed.

From the static yield function, $F = F(\sigma_{ij}, K_\alpha)$, we obtain

$$\dot{F} = \frac{\partial F}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial F}{\partial K_\alpha} \dot{K}_\alpha \tag{30}$$

With (10) and the evolution Eqs. (26) we have

$$\dot{K}_\alpha = -\Lambda D^*_{\alpha\beta} \frac{\partial G}{\partial K_\beta} \tag{31}$$

Use of (31) in (30) leads to

$$\dot{F} = \frac{\partial F}{\partial \sigma_{ij}} \dot{\sigma}_{ij} - H\Lambda \tag{32}$$

where the modulus in H is defined by

$$H = \frac{\partial F}{\partial K_\alpha} D^*_{\alpha\beta} \frac{\partial G}{\partial K_\beta} \tag{33}$$

If Hooke's incremental law (9) is inserted into (32) and use is made of the flow rule (26) for $\dot{\epsilon}_{ij}^{\text{vp}}$, we find that

$$\dot{F} = \frac{\partial F}{\partial \sigma_{ij}} D_{ijkl} \dot{\epsilon}_{kl} - A\Lambda \tag{34}$$

where the quantity A is defined by

$$A = H + \frac{\partial F}{\partial \sigma_{ij}} D_{ijkl} \frac{\partial G}{\partial \sigma_{kl}} > 0 \tag{35}$$

This quantity will be assumed to be positive. If inviscid plasticity were considered with F being the yield function and G the potential function, it then appears that the modulus H defined by (33) and the positive quantity A defined by (35) correspond to the usual definitions and that H then is the plastic modulus.

For development of viscoplasticity, $F > 0$ is required which, in turn, implies that $\Lambda > 0$, cf (24). Consider now a situation where $\dot{\epsilon}_{ij} = 0$ holds, i.e. pure relaxation is considered. Since $\Lambda > 0$ and $A > 0$, it follows from (34) that $\dot{F} < 0$; eventually, we will therefore reach the situation where $F = 0$ holds and at that state the viscoplastic development will stop.

Let us next consider the load case where the prescribed total strain rate is different from zero, i.e. $\dot{\epsilon}_{ij} \neq 0$. If this total strain rate is infinitely small, we conclude from the observation above that we will, in the limit, approach a state where $F = 0$ holds. According to (13) $f = 0$ holds during development of viscoplasticity and as (20) shows that $f = F - \bar{F}$ where both $f = 0$ and $F = 0$, it follows that $\bar{F} = 0$ holds in the limit when the total strain rate is infinitely small.

For an infinitely small total strain rate, we therefore conclude that $\bar{F} = 0$ and $F = 0$ hold in the limit. However, with the evolution laws (26) and expression (19) it follows that the only situation where $\bar{F} = 0$ holds in the limit is when Λ approaches zero. In turn, this implies that (28), which otherwise provides our expression for the unknown quantity Λ, cannot in this limit case be used to determine Λ; in fact, with Λ approaching zero, we have according to (19) that $\bar{F} \to 0$ and as also $F \to 0$, (28) simply becomes an identity.

With the observation that for the limiting case of an infinitely small total strain rate, (28) cannot be used to determine the quantity Λ, we must look for other means to cope with this problem. The solution is provided by the fact that $F = 0$ holds in the limit when the total strain rate approaches zero; this implies that also $\dot{F} = 0$ holds in the limit. From (34) we then obtain the following expression for Λ

$$\Lambda = \frac{1}{A} \frac{\partial F}{\partial \sigma_{ij}} D_{ijkl} \dot{\epsilon}_{kl} \tag{36}$$

The evolution laws (26) read

$$\dot{\epsilon}_{ij}^{\text{vp}} = \Lambda \frac{\partial G}{\partial \sigma_{ij}}; \quad \dot{\kappa}_\alpha = \Lambda \frac{\partial G}{\partial K_\alpha} \tag{37}$$

Expressions (36) and (37) hold when $F = 0$ and viscoplasticity development requires that $\Lambda > 0$; on the other hand, we know from (23) that if $F < 0$ then $\Lambda = 0$. These observations lead to the conclusion that

(36) and (37) hold always where Λ and F are subjected to the following conditions

$$\Lambda \geq 0; \quad F \leq 0; \quad \Lambda F = 0 \tag{38}$$

It appears that if Λ in (36)–(38) is renamed and now called λ and if the viscoplastic strain rate $\dot{\epsilon}_{ij}^{\mathrm{vp}}$ is renamed and now called the inviscid plastic strain rate $\dot{\epsilon}_{ij}^{\mathrm{vp}}$, then (36)–(38) become identical to those appearing in inviscid plasticity theory. Here expression (38) are the usual Kuhn–Tucker conditions and the quantity λ is usually called the plastic multiplier. The evolutions laws (37) correspond to non-associated inviscid plasticity and if the potential function G is replaced by the static yield function F, we arrive at the associated inviscid plasticity formulation, which certainly fulfils the dissipation inequality.

We have then shown that the present very general framework for viscoplasticity defined by (20) and (25) reduces in a consistent manner to classical inviscid plasticity when the total strain rate is infinitely small. In that case, the response corresponds to the so-called static stress–strain curves.

6. Hardening, perfect and softening viscoplasticity

For simplicity isothermal conditions will be considered. As discussed above, the static stress–strain response is obtained in the limit when the prescribed total strain rate is very slow. For stress states on this curve, the static yield condition is fulfilled, i.e. $F(\sigma_{ij}, K_\alpha) = 0$ and the case of uniaxial loading is shown in Fig. 1.

Assume that the loading history in some way has bought us to point A or point C and consider then the following response when the stress state is held constant. In both cases, the viscoplastic strain will increase. Starting at point A located above the rising part of the static stress–strain curve, the point (σ, ϵ) will move and eventually be located at point B on the static stress–strain curve and the total viscoplastic strain is bounded. However, starting at point C located above the falling part of the static stress–strain curve, the increasing viscoplastic strain will move the point (σ, ϵ) more and more away from the static stress–strain curve; the total viscoplastic strain in unbounded.

Noting that $F > 0$ holds both at points A and C, we may generalize these results and conclude that for a constant stress state, hardening means $\dot{F} < 0$ whereas softening means $\dot{F} > 0$. Since in the present case we have $\sigma_{ij} = 0$, we conclude from (32) that

$$H > 0 \quad \text{hardening viscoplasticity}$$

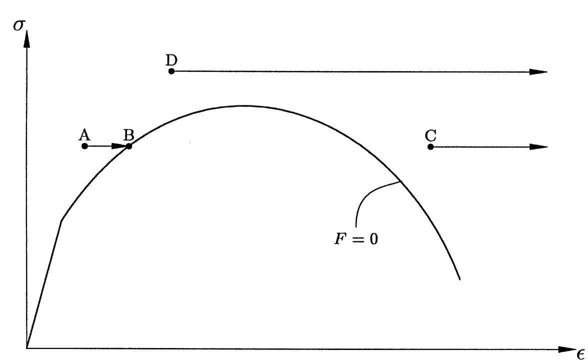

Fig. 1. Uniaxial static stress–strain curve.

$H = 0$ perfect viscoplasticity

$H < 0$ softening viscoplasticity (39)

We recall that H as defined by (33) is identical to the definition of the plastic modulus in inviscid plasticity and that (39) is identical to the definitions made in inviscid plasticity.

Finally, consider point D during a situation where the stress state is held constant. In the first place, we will approach the static yield surface, i.e. $\dot{F} < 0$ and (32) then shows that hardening occurs. At some state, $\dot{F} = 0$ will be achieved and a state of perfect viscoplasticity will emerge ($H = 0$); eventually, a state of softening viscoplasticity ($\dot{F} > 0$) will be reached and the viscoplastic strain will increase in an unbounded manner.

7. Perzyna formulation

We shall next illustrate that the restricted group of viscoplastic materials defined by (20) and (25) which possesses both a dynamic yield surface and a static yield surface also contains a variety of previously proposed viscoplastic theories as special cases. The central issue is the choice of the function $\bar{F}$ defined by (18) as well as at the choice of the static yield function $\bar{F}$, cf (20) and of the potential function G, cf (25) and (26).

We shall start with the Perzyna formulation and choose the function $\bar{F}$ defined by (18) as

$$\bar{F} = \varphi(\eta(\theta)\dot{\epsilon}^{\mathrm{vp}}_{\mathrm{eff}}) \tag{40}$$

where η is a positive parameter that may depend on the temperature θ and where $\dot{\epsilon}^{\mathrm{vp}}_{\mathrm{eff}}$ is the effective viscoplastic strain rate defined by

$$\dot{\epsilon}^{\mathrm{vp}}_{\mathrm{eff}} = \sqrt{\tfrac{2}{3}\dot{\epsilon}^{\mathrm{vp}}_{ij}\dot{\epsilon}^{\mathrm{vp}}_{ij}} \tag{41}$$

For the static yield function F, we take any convex function given by

$$F = F(\sigma_{ij}, K_\alpha, \theta) \tag{42}$$

Since $f = 0$ holds during viscoplastic development, we obtain with (20) that

$$F = \varphi(\eta(\theta)\dot{\epsilon}^{\mathrm{vp}}_{\mathrm{eff}}) \tag{43}$$

This expression is similar to (27). Referring to (18) and (19), the function φ must fulfil $\varphi \geq 0$ as well as $\varphi(0) = 0$. Let us also assume that φ is a monotonic increasing function of its argument. It then possesses an inverse function ϕ such that $\phi(\varphi(\eta(\theta)\dot{\epsilon}^{\mathrm{vp}}_{\mathrm{eff}})) = \eta(\theta)\dot{\epsilon}^{\mathrm{vp}}_{\mathrm{eff}}$. From (43) we then obtain

$$\phi(F) = \eta(\theta)\dot{\epsilon}^{\mathrm{vp}}_{\mathrm{eff}} \tag{44}$$

From the flow rule (26) and (41), we have $\dot{\epsilon}^{\mathrm{vp}}_{\mathrm{eff}} = \Lambda\sqrt{\frac{2}{3}(\partial G/\partial\sigma_{ij})(\partial G/\partial\sigma_{ij})}$. Insertion into (44) gives

$$\phi(F) = \eta(\theta)\Lambda\sqrt{\frac{2}{3}\frac{\partial G}{\partial\sigma_{ij}}\frac{\partial G}{\partial\sigma_{ij}}} \tag{45}$$

This expression is similar to (28) and it allows us to determine the unknown quantity Λ. Before this is done, we introduce the generalized McCauley bracket $\langle\ \rangle$ defined as

$$\langle \phi(F) \rangle = \begin{cases} 0 & \text{if } F \le 0 \\ \phi(F) & \text{if } F > 0 \end{cases} \tag{46}$$

In view of the loading conditions (23) and (24), we can now determine Λ from (45) and insert into the evolution Eqs. (26) to obtain

$$\dot{\epsilon}_{ij}^{\mathrm{vp}} = \frac{\langle \phi(F) \rangle}{\eta^*} \frac{\partial G}{\partial \sigma_{ij}}; \quad \dot{\kappa}_\alpha = \frac{\langle \phi(F) \rangle}{\eta^*} \frac{\partial G}{\partial K_\alpha} \tag{47}$$

where

$$\eta^* = \eta(\theta) \sqrt{\frac{2}{3} \frac{\partial G}{\partial \sigma_{ij}} \frac{\partial G}{\partial \sigma_{ij}}} \tag{48}$$

and where η^* may be viewed as a viscosity parameter. It appears that the choice of the function $\bar{F}$ given by (40) fulfils the requirement stated by (29). Therefore, associated viscoplasticity is obtained if we choose the potential function G as $G = F$ and then the formulation of Perzyna (1971) is recovered even though Perzyna did not provide any expressions for the fluxes $\dot{\kappa}_\alpha$. If, in addition, F is a convex function the dissipation inequality is evidently fulfilled. It is emphasized that here we have proved this latter statement in general and not just by considering in the spirit of Simo and Honein (1990) the limit case where $\eta^* \to 0$. For associated viscoplasticity, the Perzyna formulation is illustrated in Fig. 2.

To be strict, in the original Perzyna formulation the parameter η^* is only allowed to depend on the temperature θ, whereas η^* as defined by (48) includes the quantity $[\frac{2}{3}(\partial G/\partial \sigma_{ij})(\partial G/\partial \sigma_{ij})]^{1/2}$ which, in general, may depend on the stresses σ_{ij} and the hardening parameters K_α. However, in most cases, for instance, when the potential function G is chosen in terms of the von Mises, Drucker–Prager or Coulomb criterion the quantity $(\partial G/\partial \sigma_{ij})(\partial G/\partial \sigma_{ij})$ becomes a constant and the present formulation coincides exactly with the original Perzyna model.

To substantiate that the present formulation seems to be the most natural Perzyna formulation, we next demonstrate that the strict Perzyna formulation, i.e. $\eta^* = \eta(\theta)$ may be obtained within the present concept, but this will always lead to a format that, per definition, becomes non-associated. For this purpose, we observe that $\bar{F}$ defined by (18) in general depends also on σ_{ij} and K_α. Since also the potential function depends on these quantities, cf (25), we may choose the function $\bar{F}$ as

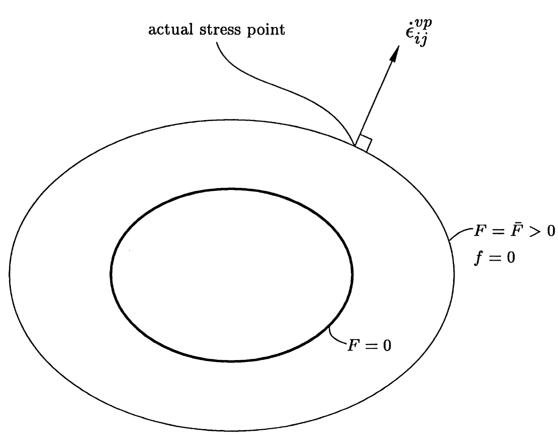

Fig. 2. Associated Perzyna formulation.

$$\bar{F} = \varphi\left(\eta(\theta)\frac{\dot{\epsilon}_{\text{eff}}^{\text{vp}}}{\sqrt{\dfrac{2}{3}\dfrac{\partial G}{\partial \sigma_{ij}}\dfrac{\partial G}{\partial \sigma_{ij}}}}\right) \tag{49}$$

For the static yield function F we choose again (42). Since $f = 0$ holds during viscoplastic development, we obtain with (20) that

$$F = \varphi\left(\eta(\theta)\frac{\dot{\epsilon}_{\text{eff}}^{\text{vp}}}{\sqrt{\dfrac{2}{3}\dfrac{\partial G}{\partial \sigma_{ij}}\dfrac{\partial G}{\partial \sigma_{ij}}}}\right) \tag{50}$$

This expression is similar to (27). As previously, we have $\dot{\epsilon}_{\text{eff}}^{\text{vp}} = \Lambda[\frac{2}{3}(\partial G/\partial\sigma_{ij})(\partial G/\partial\sigma_{ij})]^{1/2}$ and insertion into (50) gives

$$F = \varphi(\eta(\theta)\Lambda) \tag{51}$$

This expression is similar to (28). Introducing again the function ϕ, that is the inverse function to φ, we get

$$\phi(F) = \eta(\theta)\Lambda \tag{52}$$

Solving for Λ and insertion into the evolution laws (26) we are then with (46) led to

$$\dot{\epsilon}_{ij}^{\text{vp}} = \frac{\langle\phi(F)\rangle}{\eta}\frac{\partial G}{\partial \sigma_{ij}}; \quad \dot{\kappa}_\alpha = \frac{\langle\phi(F)\rangle}{\eta}\frac{\partial G}{\partial K_\alpha} \tag{53}$$

where $\eta = \eta(\theta)$; this formulation corresponds exactly to the original Perzyna model. However, an important observation is that this format never can be associative even when $G = F$. This is a consequence of the choice of $\bar{F}$ given by (49), which does not fulfil the requirement stated in (29). We therefore conclude that (47) and (48) seem to be the most natural Perzyna formulation since it includes the associativity as a special case.

Finally, it may be of interest to evaluate the situation where the parameter $\eta \rightarrow 0$. It then follows from (40) or (49) that $\bar{F} = \varphi(0)$, i.e. $\bar{F} = 0$, cf (19). Since $f = 0$ holds during viscoplasticity and as $f = F - \bar{F}$, if follows that $F = 0$, but this is precisely the requirement for having obtained inviscid plasticity. Therefore, when $\eta \rightarrow 0$ the limit of inviscid plasticity is achieved. Perzyna viscoplasticity therefore reduces to inviscid plasticity if either $\eta \rightarrow 0$ or $\dot{\epsilon}_{ij} \rightarrow 0$.

8. Odquist creep

Let us next derive the classic Odquist (1936) creep model to model secondary creep; the Odquist model represents the multiaxial formulation of Norton creep. Assume an associated formulation with no internal parameters, i.e.

$$G = F, \quad \kappa_\alpha = 0, \quad \text{i.e.} \quad K_\alpha = 0 \quad \text{and} \quad F = F(\sigma_{ij}, \theta) \tag{54}$$

Choose F as

$$F = \sigma_{\text{eff}} \quad \text{where } \sigma_{\text{eff}} = (\tfrac{3}{2}s_{kl}s_{kl})^{1/2} \tag{55}$$

and $s_{ij}=\sigma_{ij}-\delta_{ij}\sigma_{kk}/3$ denote the deviatoric stresses. With these expressions, the Perzyna formulation (47) and (48) reduces to

$$\dot{\epsilon}_{ij}^{\mathrm{vp}}=\frac{3}{2}\frac{\langle\phi(\sigma_{\mathrm{eff}})\rangle}{\eta(\theta)\sigma_{\mathrm{eff}}}s_{ij} \tag{56}$$

For the function ϕ, we may select

$$\phi(\sigma_{\mathrm{eff}})=C\sigma_{\mathrm{eff}}^{n} \tag{57}$$

where C is a positive constant and $n\geq 1$. Then (56) takes the form

$$\dot{\epsilon}_{ij}^{\mathrm{vp}}=\frac{3}{2}C\frac{\sigma_{\mathrm{eff}}^{n-1}}{\eta(\theta)}s_{ij} \tag{58}$$

It appears that we have recovered the secondary creep model of Odquist (1936). In the literature, (58) is often attributed to Bodner and Partom (1972). Moreover, if we in (58) choose $1/\eta(\theta)=\exp(-Q/(R\theta))$ where R is the universal gas constant and Q is the activation energy, the influence of absolute temperature is modelled via the exponential Arrhenius expression.

Since we have an associated formulation, the dissipation inequality is certainly fulfilled. However, this statement is easily checked. With (8), (54) and (58) and taking advantage of (55) we obtain $\gamma=\sigma_{ij}\dot{\epsilon}_{ij}^{\mathrm{vp}}=C\sigma_{\mathrm{eff}}^{n+1}/\eta$ which certainly is non-negative.

The Odquist model does not involve a purely elastic region; even so the present formulation that works with a static yield surface contains the Odquist model as a special case. The explanation for this apparent contradiction is that the static yield function F defined by (55) always fulfils $F\geq 0$, i.e. no purely elastic response occurs, cf (23) and (24).

9. Multiple yield and potential functions

Previously, we assumed that only one smooth yield and one smooth potential surface existed. Let us next consider the situation where multiple yield and potential surfaces meet at a corner. For inviscid plasticity, the associated case has been treated by Koiter (1953, 1960) and later contributions given by Mandel (1965), Hill (1966), Sewell (1973, 1974) and Simo et al. (1988). Treatment of the more general case of non-associated inviscid plasticity can be found in Ottosen and Ristinmaa (1996). First, we shall consider associated viscoplasticity. For the state in question, let $F_{\max}$ denote the total number of dynamic yield surfaces that meet at a corner, i.e.

$$f^{I}=f^{I}(\sigma_{ij},K_{\alpha},\dot{\epsilon}_{ij}^{\mathrm{vp}(J)},\dot{\kappa}_{\alpha}^{(J)},\theta)\quad I,J=1,2,\ldots,F_{\max} \tag{59}$$

where we denote $\dot{\epsilon}_{ij}^{\mathrm{vp}(J)}$ and $\dot{\kappa}_{\alpha}^{(J)}$ as the fluxes associated with the dynamic yield surface J; these fluxes will be defined later on and we note that the format (59) allows f^{I} to depend not only on the fluxes associated with the particular yield surface in question, but also on fluxes from other yield surfaces. Each of the individual dynamic yield surfaces is assumed to be smooth. Similar to (13) we have

$$f^{I}\leq 0 \tag{60}$$

and $f^{I}>0$ is not allowed.

To derive the associated viscoplastic formulation, use is again made of the postulate of maximum dissipation where the dissipation, as previously, is given by (8), i.e. $\gamma\equiv\sigma_{ij}\dot{\epsilon}_{ij}^{\mathrm{vp}}+K_{\alpha}\dot{\kappa}_{\alpha}$. We are then faced with the following problem: for given fluxes $\dot{\epsilon}_{ij}^{\rm vp}$ and $\dot{\kappa}_\alpha$ and temperature θ, find those stresses σ_{ij} and conjugated forces K_α that minimize the quantity $-\gamma$ subjected to the constraint expressed by (60). Following, for instance, Luenberger (1984) p. 314, we are then led to

$$\dot{\epsilon}_{ij}^{\rm vp} = \sum_{I=1}^{F_{\max}} \Lambda^I \frac{\partial f^I}{\partial \sigma_{ij}}; \quad \dot{\kappa}_\alpha = \sum_{I=1}^{F_{\max}} \Lambda^I \frac{\partial f^I}{\partial K_\alpha} \tag{61}$$

Note that the summation convention is not used when capital letters or Greek letters are used as superscripts. Moreover, the minimization problem leads to the Kuhn–Tucker relations expressed by

$$\Lambda^I \geq 0; \quad \Lambda^I f^I = 0 \quad \text{for all } I = 1, 2, \ldots, F_{\max} \tag{62}$$

and the functions f^I are required to be convex and smooth. We shall define the fluxes associated with yield surface f^I by

$$\dot{\epsilon}_{ij}^{{\rm vp}(I)} = \Lambda^I \frac{\partial f^I}{\partial \sigma_{ij}}; \quad \dot{\kappa}_\alpha^{(I)} = \Lambda^I \frac{\partial f^I}{\partial K_\alpha} \tag{63}$$

These fluxes therefore, define the contribution from yield function I to the evolution laws in (61).

To derive a non-associated formulation, let the potential functions meeting at a corner be defined by

$$g^\Phi = g^\Phi(\sigma_{ij}, K_\alpha, \dot{\bar{\epsilon}}_{ij}^{{\rm vp}(\Theta)}, \dot{\bar{\kappa}}_\alpha^{(\Theta)}, \theta) \quad \Phi, \Theta = 1, 2, \ldots, G_{\max} \tag{64}$$

where the total number of smooth potential surfaces meeting at the corner is denoted by $G_{\max}$ and $\dot{\bar{\epsilon}}_{ij}^{{\rm vp}(\Theta)}$, $\dot{\bar{\kappa}}_\alpha^{(\Theta)}$ are fluxes to be defined in a moment. Moreover, we have indicated that g^Φ may depend on all $G_{\max}$ fluxes. Apparently, this will cause problems for cases when $F_{\max} \neq G_{\max}$, cf (59). To solve this obstacle, the following evolution laws are postulated

$$\left.\begin{aligned} \dot{\epsilon}_{ij}^{\rm vp} &= \sum_{I=1}^{F_{\max}} \dot{\epsilon}_{ij}^{{\rm vp}(I)}; \quad \dot{\epsilon}_{ij}^{{\rm vp}(I)} = \Lambda^I \sum_{\Phi=1}^{G_{\max}} \mu^{I\Phi} \frac{\partial g^\Phi}{\partial \sigma_{ij}} \\ \dot{\kappa}_\alpha &= \sum_{I=1}^{F_{\max}} \dot{\kappa}_\alpha^{(I)}; \quad \dot{\kappa}_\alpha^{(I)} = \Lambda^I \sum_{\Phi=1}^{G_{\max}} \mu^{I\Phi} \frac{\partial g^\Phi}{\partial K_\alpha} \end{aligned}\right. \quad \Lambda^I \geq 0 \tag{65}$$

together with the constraints expressed by (60) and (62). In (65), $\dot{\epsilon}_{ij}^{{\rm vp}(I)}$ again defines the contribution of viscoplastic strain rate from the yield surface no. I. Moreover, the matrix $\mu^{I\Phi}(F_{\max} \times G_{\max})$ has been introduced. Since the quantity Λ^I is related to the corresponding f^I, cf (60) and (62), this matrix controls how much a potential function should contribute to a certain yield function. Expressions (65) may be re-cast into

$$\left.\begin{aligned} \dot{\epsilon}_{ij}^{\rm vp} &= \sum_{\Phi=1}^{G_{\max}} \dot{\bar{\epsilon}}_{ij}^{{\rm vp}(\Phi)}; \quad \dot{\bar{\epsilon}}_{ij}^{{\rm vp}(\Phi)} = \bar{\Lambda}^\Phi \frac{\partial g^\Phi}{\partial \sigma_{ij}} \\ \dot{\kappa}_\alpha &= \sum_{\Phi=1}^{G_{\max}} \dot{\bar{\kappa}}_\alpha^{(\Phi)}; \quad \dot{\bar{\kappa}}_\alpha^{(\Phi)} = \bar{\Lambda}^\Phi \frac{\partial g^\Phi}{\partial K_\alpha} \end{aligned}\right. \quad \bar{\Lambda}^\Phi = \sum_{I=1}^{F_{\max}} \Lambda^I \mu^{I\Phi} \tag{66}$$

where $\dot{\bar{\epsilon}}_{ij}^{{\rm vp}(\Phi)}$ and $\dot{\bar{\kappa}}_\alpha^{(\Phi)}$ denote the contribution of viscoplastic strain rate and the internal variable flux, respectively, from the potential function no. Φ. With the equivalent expressions (65) and (66), both definition (59) and (64) are valid expressions. The non-associated formulation (65) reduces to the associated formulation (61) if $G_{\max}=F_{\max}$, $g^I=f^I$ and $\mu^{I\Phi}=\delta^{I\Phi}$, where $\delta^{I\Phi}$ is the generalized Kronecker delta.

We may mention that instead of the format given by (65) and (66), one may adopt a formulation similar to that used by Ottosen and Ristinmaa (1996) for inviscid plasticity, but for viscoplasticity the present approach turns out to be more simple.

The non-associated formulation (65) fulfils the dissipation inequality (8) in the following situation: for given fluxes and temperature, let each potential function g^Φ be a smooth and convex function in the σ_{ij}, K_α-space. For each Φ-value, the expression $g^\Phi=C^\Phi=$constant then describes a surface in that space. If the value of g^Φ at the origin of the space is less than $C^\Phi=$constant, then it follows directly that formulation (65) fulfils the dissipation inequality (8), cf Eringen (1975).

For smooth surfaces, we saw that the condition $f=0$ enables one to determine the multiplier Λ. In a similar fashion, for multiple surfaces meeting at a corner, the conditions $f^I=0$, where $I=1, 2, \ldots, F_{\max}$, will enable us to determine $F_{\max}$ multipliers Λ^I. Now, consider (65) written as

$$\dot{\epsilon}_{ij}^{\mathrm{vp}(I)}=\Lambda^I a_{ij}^I \quad \text{where } a_{ij}^I=\sum_{\Phi=1}^{G_{\max}}\mu^{I\Phi}\frac{\partial g^\Phi}{\partial\sigma_{ij}} \tag{67}$$

From (67b), it follows that if $F_{\max}<G_{\max}$ we can have $a_{ij}^I=0$ for $\partial g^\Phi/\partial\sigma_{ij}\neq 0$, i.e. non-trivial solutions. The unacceptable implication is that even if $\Lambda^I\neq 0$, we may have $\dot{\epsilon}_{ij}^{\mathrm{vp}(I)}=0$. It then follows that we must require that

$$F_{\max}\geq G_{\max} \tag{68}$$

It is of interest that condition (68) corresponds to the conclusion arrived at for inviscid corner plasticity, cf Ottosen and Ristinmaa (1996).

Similar to the situation for smooth surfaces, cf (20) and (25), we shall consider a restricted group of materials that implies the existence of static yield functions. Similar to (18), we therefore choose functions $\bar{F}^I$ with the following properties

$$\bar{F}^I(\sigma_{ij}, K_\alpha, \dot{\epsilon}_{ij}^{\mathrm{vp}(J)}, \dot{\kappa}_\alpha^{(J)}, \theta)\geq 0 \tag{69}$$

and

$$\bar{F}^I(\sigma_{ij}, K_\alpha, \dot{\epsilon}_{ij}^{\mathrm{vp}(J)}=0, \dot{\kappa}_\alpha^{(J)}=0, \theta)=0;\quad \bar{F}^I(\sigma_{ij}, K_\alpha, \dot{\epsilon}_{ij}^{\mathrm{vp}(J)}\neq 0, \dot{\kappa}_\alpha^{(J)}\neq 0, \theta)>0 \tag{70}$$

Moreover, we choose the functions $F^I(\sigma_{ij}, K_\alpha, \theta)$ and express the dynamic yield function f^I as

$$f^I(\sigma_{ij}, K_\alpha, \dot{\epsilon}_{ij}^{\mathrm{vp}(J)}, \dot{\kappa}_\alpha^{(J)}, \theta)=F^I(\sigma_{ij}, K_\alpha, \theta)-\bar{F}^I(\sigma_{ij}, K_\alpha, \dot{\epsilon}_{ij}^{\mathrm{vp}(J)}, \dot{\kappa}_\alpha^{(J)}, \theta) \tag{71}$$

From (70) and (71), it then follows that

$$f^I(\sigma_{ij}, K_\alpha, \dot{\epsilon}_{ij}^{\mathrm{vp}(J)}=0, \dot{\kappa}_\alpha^{(J)}=0, \theta)=F^I(\sigma_{ij}, K_\alpha, \theta) \tag{72}$$

Moreover, (69) and (71) results in

$$f^I\leq F^I \tag{73}$$

The potential functions will be chosen as

$$g^\Phi=G^\Phi(\sigma_{ij}, K_\alpha, \theta) \tag{74}$$

In general, we have $f^I = 0$ during viscoplastic loading. Then by combining (71) and (65) we find

$$F^I(\sigma_{ij}, K_\alpha, \theta) = \bar{F}^I\left(\sigma_{ij}, K_\alpha, \Lambda^J \sum_{\Phi=1}^{G_{\max}} \mu^{J\Phi} \frac{\partial G^\Phi}{\partial \sigma_{ij}}, \Lambda^J \sum_{\Phi=1}^{G_{\max}} \mu^{J\Phi} \frac{\partial G^\Phi}{\partial K_\alpha, \theta}\right), \quad I = 1, 2, \ldots, F_{\max} \tag{75}$$

during viscoplastic loading. Since a number of $F_{\max}$ dynamic yield functions have been assumed to be active, (75) comprises $F_{\max}$ equations which allow us to calculate the total $F_{\max}$ numbers of Λ^J values. Expression (75) is similar to (28) valid for smooth surfaces.

Similar to the discussion related to (18)–(20), it is easily shown that formulation (69)–(71) implies the F^I becomes the static yield surfaces and that viscoplastic loading or elastic loading can be evaluated in a form similar to (23) and (24).

10. Perzyna corner viscoplasticity

Let us next generalize the Perzyna formulation (47) and (48) so that it holds at a corner. We shall consider the general case where the number of yield surfaces and potential surfaces differ, but due to (68) we have $F_{\max} \geq G_{\max}$.

Similar to (41) we define the effective viscoplastic strain rate corresponding to yield surface I by

$$\dot{\epsilon}_{\text{eff}}^{\text{vp}(I)} = \sqrt{\frac{2}{3}\dot{\epsilon}_{ij}^{\text{vp}(I)}\dot{\epsilon}_{ij}^{\text{vp}(I)}} = \Lambda^I \sqrt{\frac{2}{3}\sum_{\Theta=1}^{G_{\max}} \mu^{I\Theta} \frac{\partial G^\Theta}{\partial \sigma_{ij}} \sum_{\Phi=1}^{G_{\max}} \mu^{I\Phi} \frac{\partial G^\Phi}{\partial \sigma_{ij}}} \tag{76}$$

Then in a manner similar to (40), we next choose the functions $\bar{F}^{(I)}$ defined by (69) as

$$\bar{F}^I = \varphi^I\left(\sum_{J=1}^{F_{\max}} \eta^{IJ}(\theta)\dot{\epsilon}_{\text{eff}}^{\text{vp}(J)}\right) \tag{77}$$

where η^{IJ} denotes a matrix with size $(F_{\max} \times F_{\max})$, which may depend on the temperature. This matrix determines how the various $\dot{\epsilon}_{\text{eff}}^{\text{vp}(I)}$-values influence the development of $\bar{F}^I$. During viscoplastic development, we have that $f^I = 0$, which with (71) and (77) yields

$$F^I = \varphi^I\left(\sum_{J=1}^{F_{\max}} \eta^{IJ} \Lambda^J \sqrt{\frac{2}{3}\sum_{\Theta=1}^{G_{\max}} \mu^{J\Theta} \frac{\partial G^\Theta}{\partial \sigma_{ij}} \sum_{\Phi=1}^{G_{\max}} \mu^{J\Phi} \frac{\partial G^\Phi}{\partial \sigma_{ij}}}\right) \tag{78}$$

where advantage was taken of (76). This expression is similar to (75) and it provides the determination of all Λ^I-values. To determine these Λ^I-values explicitly, we assume similar to (43) that

$$\varphi^I(0) = 0 \quad \text{and} \quad \varphi^I \text{ are monotonic increasing functions} \tag{79}$$

Each φ^I-function therefore possesses an inverse function ϕ^I, i.e. (78) implies that

$$\phi^I(F^I) = \sum_{J=1}^{F_{\max}} \eta^{IJ} \Lambda^J \sqrt{\frac{2}{3}\sum_{\Theta=1}^{G_{\max}} \mu^{J\Theta} \frac{\partial G^\Theta}{\partial \sigma_{ij}} \sum_{\Phi=1}^{G_{\max}} \mu^{J\Phi} \frac{\partial G^\Phi}{\partial \sigma_{ij}}} \tag{80}$$

In order to be able to derive a format where this non-homogeneous equation system allows a unique

Λ^I-solution, we must require the matrix η^{IJ} to be non-singular. It therefore possesses an inverse $\tilde{\eta}^{KI}$ such that

$$\sum_{I=1}^{F_{\max}} \tilde{\eta}^{KI}\eta^{IJ} = \delta^{KJ} \tag{81}$$

where δ^{KJ} denotes a generalized Kronecker delta. Use of (81) in (80) gives the following Λ^I-values

$$\Lambda^J = \sum_{I=1}^{F_{\max}} \tilde{\eta}^{JI} \frac{\phi^I(F^I)}{\sqrt{\frac{2}{3}\sum_{\Theta=1}^{G_{\max}} \mu^{J\Theta}\frac{\partial G^{\Theta}}{\partial \sigma_{ij}} \sum_{\Phi=1}^{G_{\max}} \mu^{J\Phi}\frac{\partial G^{\Phi}}{\partial \sigma_{ij}}}} \tag{82}$$

This expression corresponds to (45) valid for smooth surfaces.

10.1. Independent hardening

Let us consider the particular case of independent hardening, i.e. hardening of one yield surface does not influence the other yield surfaces. Moreover, we shall consider the general non-associated situation, i.e. $F_{\max} \geq G_{\max}$. Referring to (77) independent hardening can be modelled by assuming that

$$\eta^{IJ} = \eta^I \delta^{IJ} \tag{83}$$

Then (77) takes the form $\bar{F}^I = \varphi^I(\eta^I \dot{\epsilon}_{\text{eff}}^{\text{vp}(I)})$, i.e. only $\dot{\epsilon}_{\text{eff}}^{\text{vp}(I)}$ will influence the corresponding yield function f^I, cf also (71). This implies that the specific dynamic yield surface hardens in a manner that is independent on the hardening of the remaining dynamic yield surfaces. Assumption (83) may therefore be viewed as an assumption of independent hardening of the dynamic yield surfaces. With (83), (82) takes the form

$$\Lambda^I = \frac{\langle \phi^I(F^I) \rangle}{\eta^{*I}} \tag{84}$$

where

$$\eta^{*I} = \eta^I(\theta)\sqrt{\frac{2}{3}\sum_{\Theta=1}^{G_{\max}} \mu^{I\Theta}\frac{\partial G^{\Theta}}{\partial \sigma_{ij}} \sum_{\Phi=1}^{G_{\max}} \mu^{I\Phi}\frac{\partial G^{\Phi}}{\partial \sigma_{ij}}} \tag{85}$$

Taking advantage of (84) in (65) the most general form of the evolution laws for independent hardening Perzyna viscoplasticity becomes

$$\dot{\epsilon}_{ij}^{\text{vp}} = \sum_{I=1}^{F_{\max}} \left(\frac{\langle \phi^I(F^I) \rangle}{\eta^{*I}} \sum_{\Phi=1}^{G_{\max}} \mu^{I\Phi} \frac{\partial G^{\Phi}}{\partial \sigma_{ij}} \right)$$

$$\dot{\kappa}_{\alpha} = \sum_{I=1}^{F_{\max}} \left(\frac{\langle \phi^I(F^I) \rangle}{\eta^{*I}} \sum_{\Phi=1}^{G_{\max}} \mu^{I\Phi} \frac{\partial G^{\Phi}}{\partial K_{\alpha}} \right) \tag{86}$$

Let us finally specialize to the important situation where $F_{\max} = G_{\max}$ and $\mu^{I\Phi} = \delta^{I\Phi}$. We then obtain

$$\dot{\epsilon}_{ij}^{\mathrm{vp}} = \sum_{I=1}^{F_{\max}} \frac{\langle \phi^I(F^I) \rangle}{\eta^{*I}} \frac{\partial G^I}{\partial \sigma_{ij}}; \quad \dot{\kappa}_\alpha = \sum_{I=1}^{F_{\max}} \frac{\langle \phi^I(F^I) \rangle}{\eta^{*I}} \frac{\partial G^I}{\partial K_\alpha} \tag{87}$$

It is recalled that the notation in (87) should be understood such that, for instance

$$\dot{\epsilon}_{ij}^{\mathrm{vp}} = \frac{\langle \phi^1(F^1) \rangle}{\eta^{*1}} \frac{\partial G^1}{\partial \sigma_{ij}} + \frac{\langle \phi^2(F^2) \rangle}{\eta^{*2}} \frac{\partial G^2}{\partial \sigma_{ij}} + \cdots + \frac{\langle \phi^{F_{\max}}(F^{F_{\max}}) \rangle}{\eta^{*F_{\max}}} \frac{\partial G^{F_{\max}}}{\partial \sigma_{ij}} \tag{88}$$

With (87), it appears that we have obtained a generalization of the original Perzyna model so that it holds even for corner viscoplasticity. If an associated formulation is adopted, i.e. $G^I = F^I$, the formulation follows directly from the postulate of maximum dissipation and the dissipation inequality (8) is then certainly fulfilled, when F^I are smooth and convex functions. A flow rule analogous to (88) was already proposed by Prager (1961) for viscoplastic solid, cf also Zarka (1972).

Finally, let us meet the argument stated by Simo et al. (1988) against the flow rule of the type (88) and often referred to in the literature as an objection against corner Perzyna viscoplasticity. Consider associated viscoplasticity and the case where a corner is formed by two intersecting yield surfaces as shown in Fig. 3. Moreover, consider the case of a non-hardening material model, i.e. no internal variables appear in the model. For inviscid plasticity, assuming that the current stress state is given by σ_{ij}, cf Fig. 3, it then follows that the inviscid corner region is formed by the tensors $D_{ijkl}\,\partial F^1/\partial \sigma_{kl}$ and $D_{ijkl}\,\partial F^2/\partial \sigma_{kl}$,, e.g. Simo et al. (1988), Ottosen and Ristinmaa (1996). Consider now viscoplasticity and the stress state $\bar{\sigma}_{ij}$ indicated in Fig. 3; evidently this stress point is located outside the inviscid corner region, but inside the viscoplastic corner region. The argument of Simo et al. (1988) is based on the conception that in the limit when $\eta^{*1} \to 0$ and $\eta^{*2} \to 0$ both yield surfaces are active since $F^1 > 0$ and $F^2 > 0$; as a consequence (88) should predict the return path $\bar{\sigma}_{ij} \to \sigma_{ij}$. This is clearly in contradiction with the inviscid solution, which is σ'_{ij}.

Analytically, it is easily shown that $\eta^{*1} \to 0$ and $\eta^{*2} \to 0$ indeed implies the inviscid solution. From (78) with $\eta^{IJ} \to 0$ and property (79), it follows that $F^I \to 0$. In the limit we therefore have $F^I = 0$ and thereby $\dot{F}^I = 0$. Similar to the previous discussion of the limit case of inviscid plasticity for smooth surfaces, we are then led exactly to the expressions that control inviscid plasticity and thereby also to

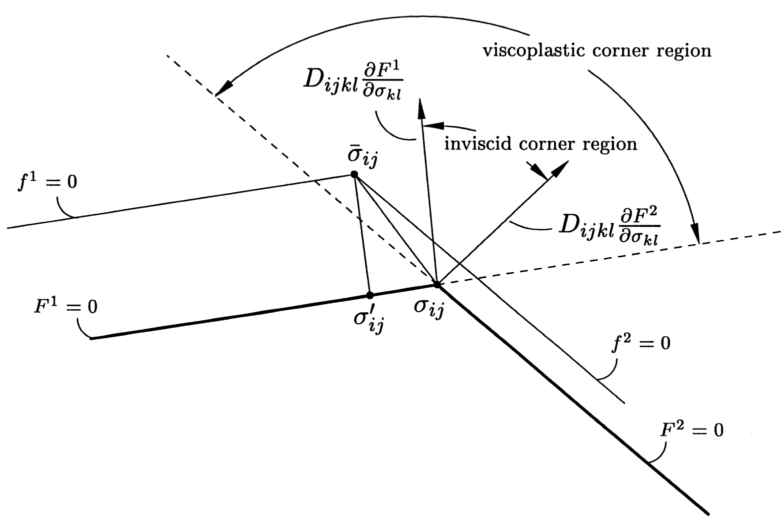

Fig. 3. Return path $\bar{\sigma}_{ij} \to \sigma_{ij}$ when $\eta^{*1} \to 0$ and $\eta^{*2} \to 0$ according to Simo et al. (1988).

the corresponding loading/unloading criteria. In the present case where corner inviscid plasticity is approached, we refer to Ottosen and Ristinmaa (1996) for further details.

However, since the geometrical argument of Simo et al. (1988), see Fig. 3, is frequently referred to in the literature, we shall also illustrate geometrically that inviscid corner plasticity is obtained in the limit when $\eta^{*1} \to 0$ and $\eta^{*2} \to 0$. Referring again to Fig. 3, it is observed that this problem formulation is not well defined, as no information is provided on how the stress point $\bar{\sigma}_{ij}$ is obtained. We therefore define the problem in the following concise fashion: we start out from the corner position σ_{ij} shown in Fig. 3. The strain rate $\dot{\epsilon}_{ij}$ and time increment Δt are considered as given and fixed and we then evaluate the response when $\eta^{*1} \to 0$ and $\eta^{*2} \to 0$.

In order to simplify the discussion, we assume isothermal conditions and associated viscoplasticity. In the case of two yield functions, (88) can with (9) be written as

$$\dot{\sigma}_{ij} = \dot{\sigma}_{ij}^{\mathrm{e}} - \dot{\sigma}_{ij}^{\mathrm{vp}(1)} - \dot{\sigma}_{ij}^{\mathrm{vp}(2)} \tag{89}$$

where

$$\dot{\sigma}_{ij}^{\mathrm{e}} = D_{ijkl}\dot{\epsilon}_{kl}$$

$$\dot{\sigma}_{ij}^{\mathrm{vp}(1)} = \frac{\langle \phi^1(F^1) \rangle}{\eta^{*1}} D_{ijkl} \frac{\partial F^1}{\partial \sigma_{kl}}$$

$$\dot{\sigma}_{ij}^{\mathrm{vp}(2)} = \frac{\langle \phi^2(F^2) \rangle}{\eta^{*2}} D_{ijkl} \frac{\partial F^2}{\partial \sigma_{kl}} \tag{90}$$

The corner position is located at point H in Fig. 4, which is the start position denoted by $(\sigma_{ij})_t$. For

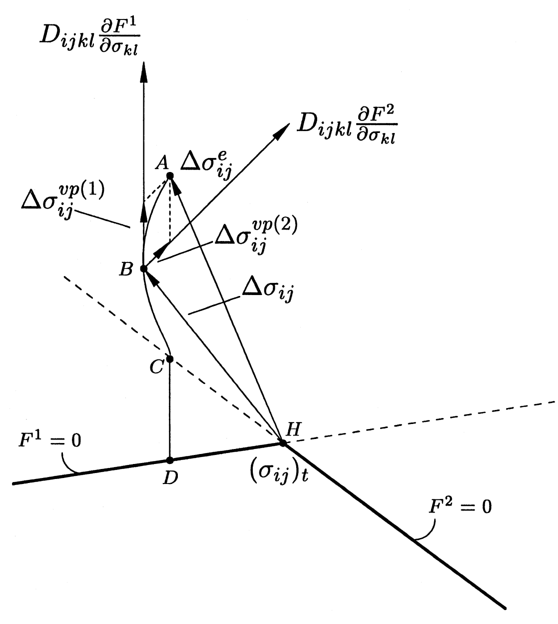

Fig. 4. Return path from stress point A when $\eta^{*1} \to 0$ and $\eta^{*2} \to 0$.

given strain rate $\dot{\epsilon}_{ij}$ and time increment Δt we want to determine the corresponding end position denoted $(\sigma_{ij})_{t+\Delta t}$. Integration of (89) gives

$$\Delta\sigma_{ij} = \Delta\sigma_{ij}^{\mathrm{e}} - \Delta\sigma_{ij}^{\mathrm{vp}(1)} - \Delta\sigma_{ij}^{\mathrm{vp}(2)} \tag{91}$$

where

$$\Delta\sigma_{ij} = (\sigma_{ij})_{t+\Delta t} - (\sigma_{ij})_t$$

$$\Delta\sigma_{ij}^{\mathrm{e}} = D_{ijkl}\dot{\epsilon}_{kl}\Delta t$$

$$\Delta\sigma_{ij}^{\mathrm{vp}(1)} = \int_t^{t+\Delta t} \frac{\langle\phi^1(F^1)\rangle}{\eta^{*1}} D_{ijkl}\frac{\partial F^1}{\partial\sigma_{kl}}\,\mathrm{d}t = \left(\frac{\langle\phi^1(F^1)\rangle}{\eta^{*1}} D_{ijkl}\frac{\partial F^1}{\partial\sigma_{kl}}\right)_{\bar{t}}\Delta t$$

$$\Delta\sigma_{ij}^{\mathrm{vp}(2)} = \int_t^{t+\Delta t} \frac{\langle\phi^2(F^2)\rangle}{\eta^{*2}} D_{ijkl}\frac{\partial F^2}{\partial\sigma_{kl}}\,\mathrm{d}t = \left(\frac{\langle\phi^2(F^2)\rangle}{\eta^{*2}} D_{ijkl}\frac{\partial F^2}{\partial\sigma_{kl}}\right)_{\bar{t}}\Delta t \tag{92}$$

Here the mean value theorem was used and the notation $(\;)_{\bar{t}}$ indicates that the quantity ( ) is evaluated at some state between t and $t+\Delta t$. Evidently, when $\Delta t \to 0$, (92) will reduce to the rate form given by (90), i.e. that the results obtained by using (92) reduces to the behaviour of (90) when $\Delta t \to 0$. As both $\dot{\epsilon}_{ij}$ and Δt are considered as fixed and given quantities, $\Delta\sigma_{ij}^{\mathrm{e}}$ is a fixed quantity. Let us assume that $\Delta\sigma_{ij}^{\mathrm{e}}$ is given as in Fig. 4, i.e. directed from the corner to point A. For the case $\eta^{*1} \to \infty$ and $\eta^{*2} \to \infty$, it follows from (91) and (92) that $\Delta\sigma_{ij} = \Delta\sigma_{ij}^{\mathrm{e}}$, i.e. the stress point will move to point A. For some finite values of η^{*1} and η^{*2} we will end up at point B. It turns out that the location of the stress point B is completely determined by the two tensors $D_{ijkl}\ \partial F^1/\partial\sigma_{kl}$, $D_{ijkl}\ \partial F^2/\partial\sigma_{kl}$, which in the present case of plane hyper-surfaces are constant tensors, and the two positive scalars $\langle\phi^1(F^1)\rangle/\eta^{*1}$ and $\langle\phi^2(F^2)\rangle/\eta^{*2}$, cf (91) and (92). An illustration of the decomposition (91) which leads to point B is also shown in Fig. 4.

Considering again the same $\dot{\epsilon}_{ij}$ and Δt, but now with smaller values of η^{*1} and η^{*2}, we will for sufficiently small η^{*1}- and η^{*2}-values end up at point C in Fig. 4. Referring to the expression for $\Delta\sigma_{ij}^{\mathrm{vp}(1)}$ and $\Delta\sigma_{ij}^{\mathrm{vp}(2)}$ given by (92), the quantities $(\;)_{\bar{t}}$ are now evaluated at some state between point H and C. It follows that at point C, we have $\Delta\sigma_{ij}^{\mathrm{vp}(1)} \neq 0$ and $\Delta\sigma_{ij}^{\mathrm{vp}(2)} = 0$, i.e. $\Delta\sigma_{ij}^{\mathrm{vp}(1)}$ goes from point C to A and CA is parallel with $D_{ijkl}\ \partial F^1/\partial\sigma_{kl}$. Eventually, considering again the same $\dot{\epsilon}_{ij}$ and Δt, but now with $\eta^{*1} \to 0$ and $\eta^{*2} \to 0$, it follows that we have obtained the return path $ABCD$ shown in Fig. 4. In the present case with plane hyper-surfaces, the tensor $D_{ijkl}\ \partial F^1/\partial\sigma_{kl}$ is constant and point D, C and A are, therefore, located on a straight line parallel with $D_{ijkl}\ \partial F^1/\partial\sigma_{kl}$. It appears that $\eta^{*1} \to 0$ and $\eta^{*2} \to 0$ leads exactly to the inviscid plasticity solution given by point D.

Turning to the case when $\Delta\sigma_{ij}^{\mathrm{e}}$ goes from H to E, cf Fig. 5, i.e. the border of the inviscid corner region. For some finite values of η^{*1} and η^{*2}, the increment leads to point G, see Fig. 5. This point can never cross the surface $F^2=0$, since $\Delta\sigma_{ij}^{\mathrm{vp}(2)} \to 0$ when $F^2 \to 0$. In conclusion, when $\eta^{*1} \to 0$ and $\eta^{*2} \to 0$, the increment will bring us to the point H, i.e. to the corner. Evidently, when the stress rate $\Delta\sigma_{ij}^{\mathrm{e}}$ is located in the inviscid corner EHE', e.g. point F, the increment will lead us to the corner H when $\eta^{*1} \to 0$ and $\eta^{*2} \to 0$. Analogous arguments hold if the stress rate $\Delta\sigma_{ij}^{\mathrm{e}}$ is located to the right of HE'.

When $\eta^{*1} \to 0$ and $\eta^{*2} \to 0$, it is concluded that the resulting stress point for viscoplastic loading ends up at the corner point, if the stress rate $\dot{\sigma}_{ij}^{\mathrm{e}}$ is located inside the inviscid corner region and the stress point ends up at one of the static yield surfaces, if the stress rate $\dot{\sigma}_{ij}^{\mathrm{e}}$ is located outside the inviscid corner region. This means that for $\eta^{*1} \to 0$ and $\eta^{*2} \to 0$ the model given by (88) reduces exactly to the inviscid model.

From the previous discussion of smooth surfaces, it is evident that when the total strain rate is infinitely small, the limit case of inviscid plasticity emerges. It is easy to show even geometrically that the correct return path is obtained also for corner viscoplasticity. Let us now consider $\Delta\epsilon_{ij}$ as a fixed and given quantity, where $\Delta\epsilon_{ij} = \dot{\epsilon}_{ij}\Delta t$. If $\dot{\epsilon}_{ij} \to 0$, we must then have $\Delta t \to \infty$. A view of expressions (91) and (92) immediately reveals that the previous arguments where $\eta^{*1} \to 0$ and $\eta^{*2} \to 0$ with fixed Δt is analogous to letting $\Delta t \to \infty$ with fixed η^{*1} and η^{*2}. Therefore, the correct inviscid corner plasticity return path is also achieved when the total strain rate is infinitely small.

10.2. Dependent hardening

If η^{IJ} is not given by (83), we have dependent hardening where development of one yield surface influences the development on other yield surfaces. As a specific type of dependent hardening, one may take

$$\eta^{IJ} = \eta_a \delta^{IJ} + \eta_b 1^{IJ} \tag{93}$$

where $1^{IJ} = 1$ for all I and J. It is emphasized that independent hardening here refers to the development of the dynamic yield surfaces. Therefore, the hardening described by (93) should not be mixed up with Budiansky and Wu (1962) type of hardening, cf Sewell (1973), for inviscid hardening, which controls the behaviour of the static yield surfaces.

With $\eta_a > 0$ and $\eta_b > 0$ (93) defines a positive definite matrix and the inverse matrix $\tilde{\eta}^{IJ}$ therefore exists. The Λ^I-values can therefore be determined from (82) and the evolution laws are then given by the previous expressions.

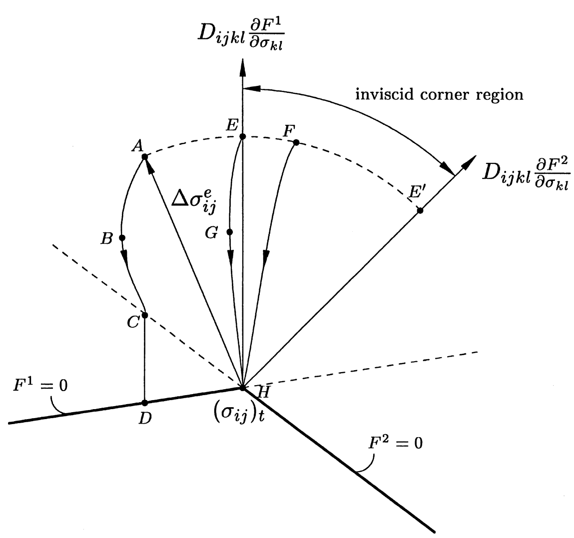

Fig. 5. Actual return paths when $\eta^{*1} \to 0$ and $\eta^{*2} \to 0$, which leads to inviscid corner plasticity.

11. Conclusions

The logical consequences of using a dynamic yield surface to model viscoplasticity within a thermodynamic framework was discussed. It turned out that the concept allows the use of the postulate of maximal dissipation and therefore of a straightforward determination of associated viscoplasticity without relying on subtle regularization and penalty techniques that only hold in the limit when viscoplasticity degenerates in inviscid plasticity. Moreover, from the definitions of the dynamic yield surface, the static yield surface emerges in a natural way.

The Perzyna viscoplastic model was derived within this concept, where it followed that the original Perzyna formulation differs with a scalar function from the formulation obeying the postulate of maximum dissipation.

It was also shown that the concept could be extended to include the case of corner viscoplasticity and a generalization of the Perzyna model was derived. Moreover, it was shown that the corner Perzyna model reduces exactly to the inviscid case when the viscosity parameters approaches zero. Therefore, the main argument often adopted in the literature against Perzyna corner viscoplasticity is removed.

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